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求解带不可微项方程的 King-Werner 迭代的收敛性

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摘要 文章给出了 King-Werner 迭代法求解带不可微项方程解的半局部收敛性定理,此结果推广了收敛性定理.

关键词 方程;解;迭代法;半局部收敛性
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1 King-Werner 迭代

设 X, Y 是 Banach 空间,$f, g: X \to Y$ 是非线性算子,$\bar{B}(x_0, R)$是 X 中以 x_0 为中心,R 为半径的闭球,设 f 在 $\bar{B}(x_0, R)$内 Frechet 可微,且满足条件:

Ⅰ) $\| f'(x) - f'(y) \| \leqslant K(r) \| x - y \|, x, y \in \bar{B}(x_0, r) \subset \bar{B}(x_0, R)$;

Ⅱ) $\| g(x) - g(y) \| \leqslant \varepsilon(R) \| x - y \|, x, y \in \bar{B}(x_0 r)$.

考虑方程 $f(x) + g(x) = 0$ (1)

的求解.由于假设条件Ⅰ)、Ⅱ)对方程(1)的要求较宽,对 $g(x)$不作可微要求,所以方程(1)的适用范围较广,为此研究方程(1)的数值求解很有意义.近年来,许多人对此类方程进行了研究,如 Zabrejko 对迭代(Zabrejko, 1980):.

$x_{n+1} = x_n - f'^{-1}(x_n)\{f(x_n) + g(x_n)\}$,和 $x_{n+1} = x_n - A^{-1}\{f(x_n) + g(x_n)\}$,

分别建立了半局部收敛性定理,其中 $A = f'(x_o)$.

本文考虑下列迭代格式:

$$x_{n+1} = x_n - f'[\frac{1}{2}(x_n + y_n)]^{-1}\{f(x_n) + g(x_n)\}, \quad (2)$$

$$y_{n+1} = x_{n+1} - f'[\frac{1}{2}(x_n + y_n)]^{-1}\{f(x_{n+1}) + g(x_{n+1})\}.$$

即为 King - Werner 迭代,由于这种迭代有许多优点,故很受人们重视(王兴华等,1982).本文对不可微项(1)方程在假设Ⅰ)、Ⅱ)下给出了迭代格式(2)的半局部收敛性定理.所得结果推广了王兴华等(1982)中的相应结论.

2 King-Werner 迭代的收敛性

设初值 x_0, y_0 满足 $\| y_0 - x_0 \| \leqslant \mu$, $\| x_1 - y_0 \| \leqslant \eta$.

$\| f'(\frac{x_0 + y_0}{2})^{-1} \| \leqslant b$.

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令 $W(t)=\int_o^t k(\tau)\mathrm{d}\tau$,假设 $1-bw(\frac{\mu}{2})>0$,

$\varphi(\mathrm{t}) = b\int_o^t W(\tau)\mathrm{d}\tau - t + [1-bw(\frac{\mu}{2})](\mu+\eta)$,

$$\psi(\mathrm{t}) = b\int_o^t \varepsilon(\tau)\mathrm{d}\tau, \tag{3}$$

$$x(\mathrm{t}) = \varphi(\mathrm{t}) + \psi(\mathrm{t}). \tag{4}$$

引理1　设 $W:\bar{B}(\mathrm{x_0},R)\to W$, W 为 Banach 空间,且满足:

$$\| W(x) - W(y) \| \leqslant h(r) \| x-y \|, x,y\in \bar{B}(\mathrm{x_0},\mathrm{r}), \tag{5}$$

则: $$\| W(x+\Delta x) - W(x) \| \leqslant \Gamma(r+\|\Delta x\|) - \Gamma(r). \tag{6}$$

其中 $x\in\bar{B}(\mathrm{x_0},\mathrm{r})$, $\|\Delta x\| \leqslant \mathrm{R}-\mathrm{r}$, $\Gamma(r)=\int_o^{\mathrm{r}} h(\tau)\mathrm{d}\tau$.

证明　设 $x\in\bar{B}(\mathrm{x_0},\mathrm{r})$, $\|\Delta x\|\leqslant R-r$,则由(5),对任何 $n\in N$, $\|W(x+\Delta x)-W(x)\|\leqslant \sum\|W(x+n^{-1}j\Delta x)-W(x+n^{-1}(j-1)\Delta x\|\leqslant\sum h(r+n^{-1}j\|\Delta x\|)n^{-1}\|\Delta x\|$. 当 $n\to\infty$ 时,由 Riemann 积分定义可得(6).

用 $\bar{x}$ 表示 $x(t)$ 在 $[0,R]$ 上的极小值,$\bar{t}$ 为 $\bar{x}$ 对应的极小值点,如果 $x(R)<0$,由于 $x(t)$ 是严格单调的,那么 $x(t)$ 在区间 $[0,t]$ 上有唯一零点 t^*.

取 King-Werner 迭代的优序列为: $t_{n+1} = t_n - \varphi'(\frac{t_n+s_n}{2})x_{(t_n)}$,

$$s_{n+1} = t_{n+1} - \varphi'(\frac{t_n+s_n}{2})^{-1}x(t_{n+1}), \tag{7}$$

其中 $S_o=\mu, t_o=0$.

引理2　若 $x(R)<0$,则对任何 $n\in\{0\}\cup N$,迭代(7)都有意义,且有不等式 $0=t_o\leqslant s_o\leqslant t_1\leqslant s_1\cdots\leqslant t_n\leqslant s_n\leqslant t^*$.

证明　显然函数 $x(t)$ 在区间 $[0,t^*)$ 是正的,且易证 $\varphi'(t)$ 在区间 $[0,t^*)$ 是负的(采用反证法).

故由序列(7)知 $\{t_n\}$ 单调增加,假设 $s_n\geqslant t_n\geqslant s_{n-1}$,则从(7)有:

$$s_{n+1} = t_{n+1} - \varphi'(\frac{s_n+t_n}{2})^{-1}x(t_{n+1}) \geqslant t_{n+1}.$$

又由:

$$t_{n+1} - s_n = \{\varphi'\frac{s_{n-1}+t_{n-1}}{2})^{-1} - \varphi'(\frac{t_n+s_n}{2})^{-1}\}x(t_n) = \varphi'(\frac{s_{n-1}+t_{n-1}}{2})^{-1}\varphi'(\frac{s_n+t_n}{2})^{-1}$$
$$\{\varphi'(\frac{t_n+s_n}{2}) - \varphi'(\frac{t_{n-1}+s_{n-1}}{2})\}x_{(t_n)}.$$

而在 $[0,\mathrm{t}^*)$ 上, $x(t)\geqslant 0$, $\varphi'(t)$ 单调增加,故 $s_{n+1}\geqslant t_{n+1}\geqslant s_n$.
依归纳法及 $x(t)$, $\varphi(t)$ 在区间 $[0,t^*)$ 端点的连续性,有:

$$0 = t_o \leqslant s_o \leqslant t_1 \leqslant s_1 \leqslant \cdots \leqslant t_n \leqslant s_n \leqslant t^*.$$

综上所述,更进一步得如下定理.

定理1　若 $x(R)<0$,则对任何 $n\in\{0\}\cup N$,迭代(2)都有意义,收敛于方程(1)在 $\bar{B}(x_o,t^*)$ 内的唯一零点 x^*,且有估计:

$$\| y_n - x_n \| \leqslant s_n - t_n, \| x^* - x_n \| \leqslant t^* - t_n \quad , \tag{8}$$

$$\| x_{n+1} - y_n \| \leqslant t_{n+1} - s_n, \| x^* - y_n \| \leqslant t^* - s_n \quad . \tag{9}$$

证明 首先用数学归纳法证明下列结论：

1° $x_n \in \bar{B}(x_o, t_n)$; 2° $\| y_n - x_n \| \leqslant s_n - t_n$;

3° $y_n \in \bar{B}(x_o, s_n)$; 4° $\| x_{n+1} - y_n \| \leqslant t_{n+1} - s_n$.

当 $n = 0$ 时，1° - 3° 显然成立，对 4° 由：

$\| x_1 - y_0 \| \leqslant \eta = ([1 - bw(\frac{s_o}{2})](\eta + \mu))/[1 - bw(\frac{s_o}{2})] - \mu = - \varphi'(\frac{s_o}{2})^{-1} x(t_0) = t_1 - s_0$.

故 4° 对 $n = 0$ 成立 .

假定当 $n \leqslant k$ 时，1° ~ 4° 均成立，当 $n = k + 1$ 时由：

$\| x_{k+1} - x_o \| \leqslant \| x_{k+1} - y_k \| + \| y_k - x_k \| + \| x_k - x_o \| \leqslant t_{k+1} - s_k + s_k - t_k + t_k - t_o = t_{k+1}$.

结合假设 Ⅰ) 和引理 1 有：

$\| f'(\frac{x_o + y_o}{2})^{-1} \{ f'(\frac{x_n + y_n}{2}) - f'(\frac{x_o + y_o}{2}) \} \| \leqslant b \{ w(\frac{t_n + s_n}{2} - w(\frac{t_o + s_o}{2}) \} < bw(\frac{t_n + s_n}{2}) = \varphi'(\frac{t_n + s_n}{2}) + 1 < 1$.

设 $T = I + f'(\frac{x_o + y_o}{2})^{-1} \{ f'(\frac{x_n + y_n}{2}) - f'(\frac{x_o + y_o}{2}) \}$，则 $f'(\frac{x_n + y_n}{2}) = f'(\frac{x_o + y_o}{2})$. T 是逆的，且有：$\| f'(\frac{x_n + y_n}{2})^{-1} \| \leqslant - b\varphi'(\frac{t_n + s_n}{2})^{-1}$.

从而：$\| y_{k+1} - x_{k+1} \| = \| f'(\frac{x_k + y_k}{2})^{-1} \{ f(x_{k+1}) + g(x_{k+1}) \} \| \leqslant$

$\| f'(\frac{x_k + y_k}{2})^{-1} \| \{ \| f(x_{k+1}) - f(x_k) - f'(\frac{x_k + y_k}{2})(x_{k+1} - x_k) \| + \| g(x_{k+1}) - g(x_k) \| \} \leqslant$

$b\varphi'(\frac{x_k + y_k}{2})^{-1} \{ \int_0^1 \| f'[x_k + \lambda(x_{k+1} - x_k)] - f'(\frac{x_k + y_k}{2})$.

$\| x_{k+1} - x_k \| d\lambda + g(x_{k+1}) - g(x_k) \| \leqslant - b\varphi'(\frac{s_k + t_k}{2})^{-1}$

$\{ \int_0^1 \| \omega[t_k + \lambda(t_{k+1} - t_k)] - \omega(\frac{s_k + t_k}{2}) \} d\lambda (t_{k+1} - t_k) - \varphi'(\frac{s_k + t_k}{2})^{-1} \{ \varphi(t_{k+1}) - \varphi(t_k) \} =$

$- b\varphi'(\frac{s_k + t_k}{2})^{-1} \int_{t_k}^{k+1} \{ W(\lambda) - W(\frac{s_k + t_k}{2}) \} d\lambda - \psi'(\frac{s_k + t_k}{2})^{-1} \{ \psi(t_{k+1}) - \psi(t_k) \} =$

$- \varphi'(\frac{s_k + t_k}{2})^{-1} \{ \varphi(t_{k+1}) - \varphi(t_k) - \varphi'(\frac{t_k + s_k}{2})(t_{k+1} - t_k) + \varphi(t_{(k+1)} - \varphi(t_k) \} = s_{k+1} - t_{k+1}$

即当 $n = k + 1$ 时，1°, 2° 成立 .

由 $\| y_{k+1} - x_o \| \leqslant \| y_{k+1} - x_{k+1} \| + \| x_{k+1} - x_o \| \leqslant s_{k+1} - t_{k+1} + t_{k+1} - t_o = s_{k+1}$.

由(2)有：

$\| x_{k+2} - y_{k+1} \| = \| f'(\frac{x_k + y_k}{2})^{-1} \{ f(x_{k+1}) + g(x_{k+1}) \} - f'(\frac{x_{k+1} + y_{k+1}}{2})^{-1} \{ f(x_{k+1}) +$

$g(x_{k+1})\} \| \leqslant f'(\frac{x_k+y_k}{2})^{-1} \| \| f'(\frac{x_{k+1}+y_{k+1}}{2}) - f'(\frac{x_k+y_k}{2}) \| \| f'(\frac{x_{k+1}+y_{k+1}}{2})\{f(k_{k+1}) + g(x_{k+1})\} \| = t_{k+2} - s_{k+1}$ 即 $n = k+1$ 时，3°，4° 成立．

由序列的收敛性及 $\| y_{n+p} - x_n \| \leqslant \sum \| y_{i+1} - y_i \| + \| y_n - x_n \| \leqslant \sum (s_{i+1} - s_i) + (s_n - t_n) = s_{n+p} - t_n$．

令 $p \to \infty$ 有：$\| x^* - x_n \| \leqslant t^* - t_n$．

同理有 $\| x^* - y_n \| \leqslant t^* - s_n$．

当初值 $x_0 = y_0$ 时有．

推论　设初值 x_0 满足：

$\| f'(x_0)^{-1} \| \leqslant b$，$\| y_1 - x_0 \| \leqslant \eta$，则当 $x(R) \leqslant 0$ 时，对于任何 $n \in N \cup \{0\}$，迭代(2)有意义，收敛于方程(1)在 $B(x_0, t^*)$ 内唯一零点 x^* 且有 $\| x^* - x_n \| \leqslant t^* - t_n$，$\| x^* - y_n \| \leqslant t^* - s_n$．

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Convergence of King-Werner Iterative for Undifferentiable Nonlinear Equations

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Abstract　A semi-local convergence theorem and an error estimation are proved for King-Werner method, when applied to undifferentiable nonlinear operator equations. The results obtained generalse and extend the ones proved in Wang Xinghua and King. R. F.

Key words　solution; equation; iteration method; semi-localconvergence

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