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一类相似逆变换矩阵的构造

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摘要 证明并给出一种构造可对角化矩阵的相似逆变换矩阵的新方法.

关键词 特征值;特征向量;相似逆变换矩阵;对角化
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对于一个 n 阶可对角化矩阵 $\boldsymbol{A}$,曾文才(1994)已给出了一种完全无须解线性方程组、而只须利用矩阵的乘法运算便可容易地求出其相似变换矩阵 $\boldsymbol{P}$ 的方法．本文将在 $\boldsymbol{P}$ 的基础上,进一步给出构造其逆矩阵的新方法．这一方法完全不同于传统的伴随矩阵法、初等变换法等．它不仅可根据 $\boldsymbol{P}$ 的构造特点简易地求出相似逆变换矩阵 $\boldsymbol{P}^{-1}$,使 $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}=\Lambda$,而且对于常系数齐次线性微分方程组 $X'(t)=\boldsymbol{A}X(t)$,也可容易地导出向量变换 $Y(t)=\boldsymbol{P}^{-1}X(t)$或 $X(t)=\boldsymbol{P}Y(t)$,从而方便地求出通解或特解.

1 定理证明

曾文才(1994)的有关定理综合叙述如下．

定理1 设 $\lambda_1,\lambda_2,\cdots,\lambda_s$ 为 n 阶矩阵 $\boldsymbol{A}$ 的相异特征值,其重数分别为 $n_1,n_2,\cdots,n_s$ 且 $\sum_{i=1}^{s}n_i=n$,则下列条件是相互等价的

(1)$\boldsymbol{A}\sim\Lambda$;

(2)$\prod_{i=1}^{s}(\lambda_iE-\boldsymbol{A})=0$;

(3)$R(W_j)=R[\prod_{i\neq j}^{s}(\lambda_iE-\mathbf{A})]=n_j$, $(j=1,2,\cdots,s)$.

定理2 设 $\lambda_1,\lambda_2,\cdots,\lambda_s$ 为 n 阶矩阵 $\boldsymbol{A}$ 的相异特征值,其重数分别为 $n_1,n_2,\cdots,n_s$ 且 $\sum_{i=1}^{s}n_i=n$. 若 $\boldsymbol{A}\sim\Lambda$,则矩阵 $\prod_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})$的列向量组中有对应于 λ_j 的 n_j 个线性无关的特征向量$(j=1,2,\cdots,s)$.

根据上述定理,设:$w_j=\prod_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})=(\alpha_{j1},\alpha_{j2},\cdots,\alpha_{jn_j},\alpha_{jn_j+1},\cdots,\alpha_{jn})$. 因 $R(W_j)=n_j$,不妨设 $\alpha_{j1},\alpha_{j2}\cdots,\alpha_{jn_j}$线性无关,则 w_j 的后 $n-n_j$ 列可被线性表示为:

$$\alpha_{jn_j+m}=k_{j1}{}^{(m)}\alpha_{j1}+\cdots+k_{jn_j}{}^{(m)}\alpha_{jn_j}\quad,\quad(j=1,2,\cdots,s,\ m=1,2,\cdots,n-n_j).$$

于是,相似变换矩阵 $\boldsymbol{P}$ 可构造为:

$$\boldsymbol{P}=(\alpha_{11},\cdots,\alpha_{1n_1},\cdots,\alpha_{j1},\cdots,\alpha_{jn_j},\alpha_{s_1}\cdots,\alpha_{sn_s}).$$

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定理 3 在定理 2 的条件下,$\boldsymbol{A}$ 的相似逆变换矩阵

$$\boldsymbol{P}^{-1}=(\beta_{11},\cdots,\beta_{1n_1},\cdots,\beta_{j1},\cdots,\beta_{jn_j},\cdots,\beta_{s1},\cdots,\beta_{sn_s}).$$

其中: $\beta_{j1}=\dfrac{1}{\Delta_j}[1,0,\cdots,0,k_{j1}{}^{(1)},\cdots,k_{j1}{}^{(n-n_j)}]$,

$\beta_{j2}=\dfrac{1}{\Delta_j}[1,0,\cdots,0,k_{j2}{}^{(1)},\cdots,k_{j2}{}^{(n-n_j)}]$,

$\vdots$

$\beta_{jn_j}=\dfrac{1}{\Delta_j}[1,0,\cdots,0,k_{jn_j}{}^{(1)},\cdots,k_{jn_j}{}^{(n-n_j)}]$,

而 $\Delta_j=\prod\limits_{i\neq j}^{s}(\lambda_i-\lambda_j)$, $(j=1,2,\cdots,s)$.

证 一方面:$w_j=\prod\limits_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})=(\alpha_{j1},\alpha_{j2},\cdots,\alpha_{jn_j},\alpha_{jn_j+1},\cdots,\alpha_{jn})$. (1)

另一方面:

$$\begin{aligned}w_j\boldsymbol{P}=\prod_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})\boldsymbol{P}&=[\prod_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})](\alpha_{11},\cdots,\alpha_{1n_1},\cdots,\alpha_{j1},\cdots,\alpha_{jn_j},\cdots,\alpha_{s1},\cdots,\alpha_{sn_s})\\&=[(\prod_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})\alpha_{11}],\cdots[\prod_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})]\alpha_{1n_1},\cdots,[\prod_{i\neq j}^{s}[\lambda_iE-\boldsymbol{A})]\alpha_{j1},\cdots,\\&\quad[\prod_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})]\alpha_{jn_j},\cdots,[\prod_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})]\alpha_{s1},\cdots,[\prod_{i\neq j}^{s}(\lambda_iE-\boldsymbol{A})\alpha_{sn_s}],\end{aligned}$$

从而 $w_j\boldsymbol{P}=\{0,\cdots,0,[\prod\limits_{i\neq j}^{s}(\lambda_i-\lambda_j)]\alpha_{j1},\cdots,[\prod\limits_{i\neq j}^{s}(\lambda_i-\lambda_j)]\alpha_{jn_j},0,\cdots,0\}$. (2)

等式(2)两边右乘 $\boldsymbol{P}^{-1}$,得:

$$w_j=\{0,\cdots,0,[\prod_{i\neq j}^{s}(\lambda_i-\lambda_j)]\alpha_{j1},\cdots,[\prod_{i\neq j}^{s}(\lambda_i-\lambda_j)]\alpha_{jn_j},0,\cdots,0\}\times(\beta_{11},\cdots,\beta_{1n_1},\cdots,\beta_{j1},\cdots,\beta_{jn_j},\cdots,\beta_{s1},\cdots,\beta_{sn_s}).$$

即有 $w_j=[\prod\limits_{i\neq j}^{s}(\lambda_i-\lambda_j)](\alpha_{j1}\beta_{j1}+\cdots+\alpha_{jn_j}\beta_{jn_j})$. (3)

由(1)和(3)有:

$$(\alpha_{j1},\alpha_{j2},\cdots,\alpha_{jn_j},\alpha_{jn_j+1},\cdots,\alpha_{jn})=[\prod_{i\neq j}^{s}(\lambda_i-\lambda_j)](\alpha_{j1}\beta_{j1}+\cdots+\alpha_{jn_j}\beta_{jn_j}).\quad(4)$$

而 w_j 的后$(n-n_j)$列可被线性表示为:

$$\alpha_{jn_j+m}=k_{j1}{}^{(m)}\alpha_{j1}+\cdots+k_{jn_j}{}^{(m)}\alpha_{jn_j},\quad(j=1,2,\cdots,s,\ m=1,2,\cdots,n-n_j).$$

比较(4)式,得:

$\beta_{j1}=\dfrac{1}{\Delta_j}[1,0,\cdots,0,k_{j1}{}^{(1)},\cdots,k_{j1}{}^{(n-n_j)}]$,

$\beta_{j2}=\dfrac{1}{\Delta_j}[1,0,\cdots,0,k_{j2}{}^{(1)},\cdots,k_{j2}{}^{(n-n_j)}]$,

$\vdots$

$\beta_{jn_j}=\dfrac{1}{\Delta_j}[1,0,\cdots,0,k_{jn_j}{}^{(1)},\cdots,k_{jn_j}{}^{(n-n_j)}]$,

而 $\Delta_j=\prod\limits_{i\neq j}^{s}(\lambda_i-\lambda_j)$, $(j=1,2,\cdots,s)$ 得证.

注:若 w_j 的 n_j 个线无关的列向量不在或不取 w_j 前 n_j 列,则 $\boldsymbol{P}$ 的形式有所改变,相应地,$\boldsymbol{P}^{-1}$的形式也有所改变,但求法是类似的.

2 几个实例

例1 求 $A=\begin{pmatrix}1&1&0&-1\\1&1&-1&0\\0&-1&1&1\\-1&0&1&1\end{pmatrix}$ 的相似变换矩阵和相似逆变换矩阵.

解 A 的特征值 $\lambda_1=1$(二重), $\lambda_2=3$, $\lambda_3=-1$.

$$w_1=(\lambda_2E-A)(\lambda_3E-A)=\begin{pmatrix}-2&0&-2&0\\0&-2&0&-2\\-2&0&-2&0\\0&-2&0&-2\end{pmatrix}=(\alpha_{11},\alpha_{12},\alpha_{13},\alpha_{14})$$

$$=(\alpha_{11},\alpha_{12},\alpha_{11},\alpha_{12})=\alpha_{11}(1,0,1,0)+\alpha_{12}(0,1,0,1),$$

$$w_2=(\lambda_1E-A)(\lambda_3E-A)=\begin{pmatrix}2&2&-2&-2\\2&2&-2&-2\\-2&-2&2&2\\-2&-2&2&2\end{pmatrix}=(\alpha_{21},\alpha_{22},\alpha_{23},\alpha_{24})$$

$$=(\alpha_{21},\alpha_{21},-\alpha_{21},-\alpha_{21})=\alpha_{21}(1,1,-1,-1),$$

$$w_3=(\lambda_1E-A)(\lambda_2E-A)=\begin{pmatrix}2&-2&-2&2\\-2&2&2&-2\\-2&2&2&-2\\2&-2&-2&2\end{pmatrix}=(\alpha_{31},\alpha_{32},\alpha_{33},\alpha_{34})$$

$$=(\alpha_{31},\alpha_{31},-\alpha_{31},-\alpha_{31})=\alpha_{31}(1,-1,-1,1).$$

λ_1 的两个线性无关的特征向量:

$\alpha_{11}=(-2,0,-2,0)'$, $\alpha_{12}=(0,-2,0,-2)'$ (取 w_1 的前两列).

λ_2 的一个特征向量: $\alpha_{21}=(2,2,-2,-2)'$ (取 w_2 的第1列).

λ_3 的一个特征向量: $\alpha_{31}=(2,-2,-2,2)'$ (取 w_3 的第1列).

故相似变换矩阵:

$$P=(\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{31})=\begin{pmatrix}-2&0&2&2\\0&-2&2&-2\\-2&0&-2&-2\\0&-2&-2&2\end{pmatrix}.$$

$$\beta_{11}=\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}(1,0,k_{11}^{(1)},k_{11}^{(2)})=-\frac{1}{4}(1,0,1,0),$$

$$\beta_{12}=\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}(0,1,k_{12}^{(1)},k_{12}^{(2)})=-\frac{1}{4}(0,1,0,1),$$

$$\beta_{21}=\frac{1}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)}(1,k_{21}^{(1)},k_{21}^{(2)},k_{21}^{(3)})=\frac{1}{8}(1,1,-1,-1),$$

$$\beta_{31}=\frac{1}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}(1,k_{31}^{(1)},k_{31}^{(2)},k_{31}^{(3)})=\frac{1}{8}(1,-1,-1,1).$$

相似逆矩阵:

$$P^{-1}=(\beta'_{11},\beta'_{12},\beta'_{21},\beta'_{31})'=-\frac{1}{8}\begin{pmatrix}2&0&2&0\\0&2&0&2\\-1&-1&1&1\\-1&1&1&-1\end{pmatrix}.$$

例 2　求常系数线性齐次微分方程组 $X'(t)=AX(t)$ 的通解,

其中 $A=\begin{pmatrix}7&4&-1\\4&7&-1\\-4&-4&4\end{pmatrix}$, $X(t)=\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix}$, $X'(t)=\begin{pmatrix}x'_1(t)\\x'_2(t)\\x'_3(t)\end{pmatrix}$.

解　A 的特征值　$\lambda_1=3$ (二重), $\lambda_2=12$.

$$w_1=(\lambda_2E-A)=\begin{pmatrix}5&-4&1\\-4&5&1\\4&4&8\end{pmatrix}=(\alpha_{11},\alpha_{12},\alpha_{13})=(-\alpha_{12}+\alpha_{13},\alpha_{12},\alpha_{13})$$

$$=\alpha_{12}(-1,1,0)+\alpha_{13}(1,0,1),$$

$$w_2=(\lambda_1E-A)=\begin{pmatrix}-4&-4&1\\-4&-4&1\\4&4&-1\end{pmatrix}=(\alpha_{21},\alpha_{22},\alpha_{23})=(-4\alpha_{23},-4\alpha_{23},\alpha_{23})$$

$$=\alpha_{23}(-4,-4,1)$$

λ_1 的两个线性无关的特征向量: $\alpha_{12}=(-4,5,4)'$, $\alpha_{13}=(1,1,8)'$,(取 w_1 的后两列).

λ_2 的一个特征向量: $\alpha_{23}=(1,1,-1)'$, (取 w_2 的第 3 列).

故相似变换矩阵:

$$P=(\alpha_{12},\alpha_{13},\alpha_{23})=\begin{pmatrix}-4&1&1\\5&1&1\\4&8&-1\end{pmatrix}.$$

而　$\beta_{12}=\dfrac{1}{(\lambda_2-\lambda_1)}(-1,1,0)=\dfrac{1}{9}(-1,1,0)$, $\beta_{13}=\dfrac{1}{(\lambda_2-\lambda_1)}(1,0,1)=\dfrac{1}{9}(1,0,1)$,

$\beta_{23}=\dfrac{1}{(\lambda_1-\lambda_2)}(-4,-4,1)=-\dfrac{1}{9}(-4,-4,-1)=\dfrac{1}{9}(4,4,1)$,

$$P^{-1}=(\beta'_{12},\beta'_{13},\beta'_{23})'=\frac{1}{9}\begin{pmatrix}-1&1&0\\1&0&1\\4&4&-1\end{pmatrix}.$$

作变换　$X(t)=PY(t)$.　则 $X'(t)=PY'(t)$, $Y'(t)=P^{-1}APY(t)=\Lambda Y(t)$. 于是,
原方程的通解为

$$X(t)=PY(t)=\begin{pmatrix}-4&1&1\\5&1&1\\4&8&-1\end{pmatrix}\begin{pmatrix}c_1e^{3t}\\c_2e^{3t}\\c_3e^{12t}\end{pmatrix}=c_1\begin{pmatrix}-4\\5\\4\end{pmatrix}e^{3t}+c_2\begin{pmatrix}1\\1\\8\end{pmatrix}e^{3t}+c_3\begin{pmatrix}1\\1\\-1\end{pmatrix}e^{12t}.$$

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The Construction of Similarity Inverse Transformation Matrices of a Diagonalizable Matrix

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Abstract This paper proves and provides a new method for constructing the similarity inverse transformation matrices of diagonalization matrix.

Key words eigenvalues; eigenvectors; similarity inverse transformation matrices; diagonalization

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Study on the Heat Pump Dehumidifying and Thermal Airflow Drying *Flammulina valutipes*

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Abstract Heat pump dehumidifying and thermal airflow drying equipment were used to carry out drying tests on *Flammulina valutipes* by means of quadratic cross rotary regression design. The pattern of the effect of (1) mean airflow velocity, (2) material water content at the change over point from heat pump dehumidifying to thermal airflow drying and (3) temperature of thermal airflow on dehydration rate, energy consumption and rehydration rate of dried product was probed. The optimum parameters for this drying process were provided.

Key words *Flammulina valutipes*; heat pump dehumidifying and thermal airflow drying; optimum parameters

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